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Classical mechanics as a topological field theory

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ABSTRACT

In this paper we show that our recent formulation of classical mechanics via path-integrals has many features in common with Witten's topological field theories. The action of the theory is a pure *BRS*-commutator, which arises when we "gauge-fix" the symmetry of arbitrary deformations in path-space. This gauge fixing involves the Hamiltonian H and the symplectic 2-form ω on the phase space $\mathcal{M}_{2n}$. Imposing *BRS*-invariant boundary conditions, the path-integral can be used to calculate "topological" invariants of the dynamical system $(\mathcal{M}_{2n}, \omega, H)$. In this paper we restrict ourselves to the Euler number of the symplectic manifold $\mathcal{M}_{2n}$ and the Maslov indices of the dynamics. Most probably many other "topological" features of $(\mathcal{M}_{2n}, \omega, H)$ can be calculated using this path-integral. Conventional (non-topological) classsical mechanics is recovered by using boundary conditions which are not *BRS*-invariant. They give rise to expectation values which are sensitive to the "local" details of the dynamics.

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In two previous papers [1] we have given a path-integral formulation of *classical mechanics* both in its Lagrangian and in its Hamiltonian version. The operatorial contents of this theory turned out to be a generalization of the operatorial approach to classical mechanics of Koopman and von Neumann [2] . The path-integral measure was chosen in such a way that only classically allowed paths contributed to the integral. This measure could be written as $exp(i\widetilde{S})$, where the "super-action" $\widetilde{S}$ not only contains the original phase-space variables $\phi^a(t)$, but also anticommuting ghosts fields $c^a(t)$ and $\bar{c}_a(t)$, as well as a commuting auxiliary field $\lambda_a(t)$. These ghosts have interesting dynamical and geometrical properties. From the dynamical point of view their importance lies in the fact that they can be identified with the well-known Jacobi fields [3] (or "geodesic deviations") which describe the behaviour of nearby trajectories. This kind of information is needed to detect the possible chaotic behaviour of a dynamical system [4] which, in fact, was one of the main motivations for setting up a path-integral formulation of classical dynamics. From the geometrical point of view the ghosts are interesting because in a Schrödinger picture-like formulation [1] of the theory, c^a and $\bar{c}_a$ become, respectively, multiplicative and derivative operators acting on generalized density functions $\widetilde{\varrho}(\phi^a, c^a)$. In this context the ghosts c^a provide a basis in cotangent space (like the differentials $d\phi^a$), and consequently all the manipulations of exterior algebra on a symplectic manifold (Cartan calculus [5]) can be reformulated as a set of operatorial rules involving the ghosts. The exterior derivative, for instance, is realized by the conserved charge of a hidden *BRS* symmetry of $\widetilde{S}$. More generally, for any Hamiltonian system, $\widetilde{S}$ is invariant under the action of a universal *ISp(2)* symmetry group, and all its generators have a simple geometric interpretation [1] . Moreover, in ref. [6] it has been shown that, in the path-integral formulation, every conservation law of the system leads to a further graded symmetry of $\widetilde{S}$. In particular a "genuine" supersymmetry makes its appearance in relation to the energy conservation. This supersymmetry can be used to characterize ergodic systems: all systems with this supersymmetry not spontaneously broken are ergodic.

In the present paper we are going to discuss another rather surprising feature of the path-integral for classical mechanics: it gives rise to a kind of topological field theory similar to the theories recently proposed by Witten [7] . In particular, the Lagrangian $\widetilde{\mathcal{L}}$, belonging to the action $\widetilde{S}$, turns out to be a pure *BRS*-commutator. We shall see that the classical path-integral can be used to compute "topological"* invariants of any *dynamical system* $(\mathcal{M}_{2n}, \omega, H)$, where $\mathcal{M}_{2n}$ is a symplectic manifold with symplectic 2-form ω

* By "topological" we mean quantities insensitive to "small" deformations of $(\mathcal{M}_{2n}, \omega, H)$ but sensitive to "large" deformations.

and H is the Hamiltonian. The simplest example is provided by quantities which are insensitive† to ω and H and therefore do not "feel" the dynamics. They are the ordinary topological invariants of $\mathcal{M}_{2n}$, the cohomology classes $\mathbf{H}^p(\mathcal{M}_{2n}.\Re)$ or, in particular, the Euler number $\chi(\mathcal{M}_{2n})$. There is also another class of invariants which are sensitive to the dynamics, but which do not change if we slightly perturb the dynamics. A typical example of this class is the Maslov index.

To start with, we recall some of the basic features of the classical path-integral. We assume the reader to be familiar with ref. [1] and mention only a few important points. We consider a Hamiltonian system with a time-independent Hamiltonian $H(\phi^a)$ defined on a 2n-dimensional phase-space $\mathcal{M}_{2n}$ with local coordinates $\phi^a, a = 1, \cdots, 2n$. Hamilton's equations are

$$\dot{\phi}^a(t) = h^a(\phi(t)) \tag{1}$$

where $h^a \equiv \omega^{ab}\partial_b H$ is the Hamiltonian vector field generated by H (ω^{ab} is the symplectic structure [5] .) The classical path-integral describes the time-evolution of generalized phase-space density functions $\widetilde{\varrho}(\phi^a, c^a, t)$ which, in general, depends on the ghosts. If we consider $\{c^a\}$ as a basis of the cotangent space $T^*_\phi\mathcal{M}_{2n}$ at some $\phi \in \mathcal{M}_{2n}$, then (ϕ^a, c^a) are local coordinates for the cotangent bundle $T^*\mathcal{M}_{2n}$. Expanding the $\widetilde{\varrho}$ as

$$\widetilde{\varrho}(\phi^a, c^a, t) = \sum_{p=o}^{2n} \frac{1}{p!}\varrho^{(p)}_{a_1\cdots a_p}(\phi^a, t)\; c^{a_1}\cdots c^{a_p} \tag{2}$$

we see that we can interpret it as a kind of inhomogeneous differential form, with the lowest component $\varrho^{(0)} \equiv \varrho(\phi^a)$ being the usual scalar density on phase space. The time evolution of $\widetilde{\varrho}$ is given by

$$\widetilde{\varrho}(\phi^a_f, c^a_f, t_f) = \int d^{2n}\phi_i\; d^{2n}c_i\; K(\phi^a_f, c^a_f, t_f|\phi^a_i, c^a_i, t_i)\widetilde{\varrho}(\phi^a_i, c^a_i, t_i) \tag{3}$$

where the kernel K has the path-integral representation

$$K(\phi_f, c_f, t_f|\phi_i, c_i, t_i) \;=\; \int \mathcal{D}\phi\; \mathcal{D}\lambda\; \mathcal{D}c\; \mathcal{D}\bar{c}\; exp\; i\int_{t_i}^{t_f} dt\widetilde{\mathcal{L}} \tag{4}$$

† This means insensitive to both "small" and "large" deformations of ω and H.

subject to the boundary conditions

$$\begin{aligned} \phi^a(t_i) = \phi^a_i \ , \ \phi^a(t_f) = \phi^a_f \\ c^a(t_i) = c^a_i \ , \ c^a(t_f) = c^a_f \end{aligned} \tag{5}$$

The initial values of the momenta λ_a and $\bar{c}_a$ are integrated, but the path-integral is independent of the final values* . In terms of the "super-Hamiltonian"

$$\widetilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i\bar{c}_a \omega^{ac} \partial_c \partial_b H c^b \tag{6}$$

the Lagrangian $\widetilde{\mathcal{L}}$ is given by $\widetilde{\mathcal{L}} = \lambda_a \dot{\phi}^a + i\bar{c}_a \dot{c}^a - \widetilde{\mathcal{H}}$. Explicitly evaluating the path-integral (4) one finds

$$K(\phi^a_f, c^a_f, t_f | \phi^a_i, c^a_i, t_i) = \delta^{(2n)}(\phi^a_f - \phi^a_{cl}(t_f, \phi_i))\delta^{(2n)}(c^a_f - c^a_{cl}(t_f, c_i, [\phi])) \tag{7}$$

Here ϕ^a_{cl} and c^a_{cl} denote solutions of Hamilton's equation of motion (1) and of the ghost equation of motion

$$[\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H(\phi)] c^b = 0 \tag{8}$$

with initial values ϕ^a_i and c^a_i, respectively. Equation (7) clearly shows that we are doing classical mechanics rather than quantum mechanics, since a particle can propagate from an initial point to some final point only if these two points are connected by a classical trajectory. If we are not interested in the ghosts, we may integrate (7) over c^a_f and are then left with the propagation kernel for ordinary density functions $\varrho(\phi^a)$. Formally we can rewrite (3) as

$$\widetilde{\varrho}(\phi^a, c^a, t) = e^{-it\widetilde{\mathcal{H}}}\widetilde{\varrho}(\phi^a, c^a, 0) \tag{9}$$

Where $\widetilde{\mathcal{H}}$ is now a "Schrödinger" picture differential operator with λ_a replaced by $-i\frac{\partial}{\partial\phi^a} \equiv -i\partial_a$ and $\bar{c}_a$ by $\frac{\partial}{\partial c^a}$. Looking at the component fields of eq.(2) we find that in this representation $\widetilde{\mathcal{H}}$ is essentially the same as the Lie derivative along the Hamiltonian vector field $h^a(\phi^c) = \omega^{ab}\partial_b H(\phi^c)$, generated by H:

$$\widetilde{\mathcal{H}} = -il_h \tag{10}$$

where

$$l_h = h^a \partial_a + c^b(\partial_b h^a)\frac{\partial}{\partial c^a} \tag{11}$$

is the Lie-derivative operator along the vector field h^a . In refs.[1] and [5] , we studied in detail the symmetries of the action $\widetilde{S} = \int dt \widetilde{\mathcal{L}}$. Here we only mention the most important

* This is the same situation as for a Dirac field [8] .

one, namely the *BRS*-invariance of $\widetilde{S}$. The *BRS* transformations act on the fields as

$$\begin{aligned} \delta\phi^a &= \epsilon c^a \\ \delta\bar{c}_a &= i\epsilon\lambda_a \\ \delta c^a &= \delta\lambda_a = 0 \end{aligned} \tag{12}$$

They are generated by the conserved charge

$$Q = i\, c^a \lambda_a \tag{13}$$

according to $\delta(\cdot) = [\epsilon Q, (\cdot)]$ where the non-vanishing (graded) commutators are given by

$$\begin{aligned} [\phi^a, \lambda_b] = i\delta^a_b \\ [\bar{c}_b, c^a] = \delta^a_b \end{aligned} \tag{14}$$

In the Schrödinger picture Q becomes $Q = c^a\partial_a$, so that the *BRS* operator can be identified with the exterior derivative.

Before turning to the topological field theory aspects of our approach to classical mechanics (CM), we briefly mention a slight generalization of the path-integral (4) which will be useful later on. We consider N distinguishable species of particles, all of which evolve according to Hamilton's equation (1) with the *same* Hamiltonian vector field $h^a \equiv \omega^{ab}\partial_a H$. The time-evolution kernel for their generalized densities $\widetilde{\varrho}(\phi^a_1, c^a_1, \phi^a_2, c^a_2, \cdots \phi^a_N, c^a_N)^*$ is given by a product of the δ-functions in eq.(7)

$$\begin{aligned} K_N(\{\phi_{k,f}\}, \{c_{k,f}\}, t_f | \{\phi_{k,i}\}, \{c_{k,i}\}, t_i) &= \prod_{k=1}^{N} K(\phi_{k,f}, c_{k,f}, t_f | \phi_{k,i}, c_{k,i}, t_i) \\ &= \int \mathcal{D}\phi^a_k\, \mathcal{D}\lambda_{k,a} \mathcal{D}c^a_k \mathcal{D}\bar{c}_{k,a}\ exp\ i \int_{t_i}^{t_f} dt \widetilde{\mathcal{L}}_N \end{aligned} \tag{15}$$

The path-integral representation of K_N is again of the form (4) but with $\mathcal{D}\phi^a(t)\mathcal{D}\lambda_a(t)\cdots$ replaced by $\mathcal{D}\phi^a_k(t)\mathcal{D}\lambda_{ak}\cdots$, and the Lagrangian $\widetilde{\mathcal{L}}$ replaced by $\widetilde{\mathcal{L}}_N = \sum_{k=1}^N \widetilde{\mathcal{L}}(\phi^a_k, \lambda_{a,k}, c^a_k, \bar{c}_{a,k})$. Since the various particle species do not "interact", the resulting N-particle theory is simply the N-fold tensor product of the original one. The action $\widetilde{S}_N = \int dt \widetilde{\mathcal{L}}_N$ is invariant under a set of N independent *BRS* transformations of the form (12) . We shall come back to this generalization when we discuss the path-integral representation for the Maslov index.

* This function gives the "super-probability" to find a particle of species 1 at (ϕ^a_1, c^a_1), of species 2 at (ϕ^a_2, c^a_2) ,etc..

Now let us describe what our approach to CM has in common with topological field theory models [6]. The crucial observation is that the Hamiltonian $\widetilde{\mathcal{H}}$ and the Lagrangian $\widetilde{\mathcal{L}}$ are pure *BRS* variations. Using (12), or (13) with (14), one easily verifies that

$$\widetilde{\mathcal{H}} = -i\Big[Q, [\bar{Q}, H]\Big] \tag{16}$$

and

$$\widetilde{\mathcal{L}} = -i[Q, \chi] \tag{17}$$

where $\bar{Q} \equiv i\bar{c}_a\omega^{ab}\lambda_b$ is the anti-*BRS* operator [1] and where

$$\chi \equiv \bar{c}_a[\dot{\phi}^a - h^a(\phi)] \tag{18}$$

Interpreting (17) in the language of the standard Hamiltonian *BRS*- approach to gauge theories [9], one would conclude that the Lagrangian $\widetilde{\mathcal{L}}$ is completely determined by the "gauge fixing function" χ, because the gauge-fixing condition enters the Batalin-Fradkin-Vilkovisky path-integral [10] precisely in the form of a BRS-commutator $[Q, \chi]$. A natural question to ask is whether there exists a local symmetry which, upon gauge fixing, gives rise to the BRS invariance of $\widetilde{\mathcal{L}}$. This is exactly what was shown to happen for the fermionic symmetry of Witten's original theory [11]. In the present case it is not difficult to find the underlying local symmetry. Note that in eq.(18) Hamilton's equation appears as a kind of gauge-fixing condition. In fact, let us consider a theory of paths $\phi^a(t)$ in phase-space, which has an identically vanishing action $S[\phi] \equiv 0$. Clearly, for this trivial action each path is as good as any other path and so S is invariant under the transformation

$$\phi^a(t) \rightarrow \phi^a(t) + \delta\phi^a(t) \tag{19}$$

where $\delta\phi^a(t)$ is a completely arbitrary deformation of the original path. Very formally, the partition function for this theory would be

$$\mathbf{Z} = \int \mathcal{D}\phi^a(t) \tag{20}$$

i.e., it is a funcctional integral over a constant. To give a meaning to (20), let us gauge fix the local† "gauge" symmetry (19). We choose the following gauge condition:

$$G^a \equiv \dot{\phi}^a - h^a(\phi^b) \approx 0 \tag{21}$$

where $h^a \equiv \omega^{ab}\partial_b H$ is the vector field generated by some Hamiltonian H. Obviously the paths obeying the condition (21) are exactly the solutions of Hamilton's equation.

† Here and in the following "local" means local in t.

In a heuristic manner we can now apply the Faddeev-Popov trick to (20). We implement (21) by a δ-function $\delta[G^a]$ multiplied by the Faddeev-Popov determinant $det(\frac{\delta G^a}{\delta\phi^b})$. This yields

$$\mathbf{Z} = \int \mathcal{D}\phi^a(t)\ \delta\big[\dot{\phi}^a - h^a(\phi)\big]\ det\big[\delta^a_b\partial_t - \partial_b h^a(\phi)\big] \tag{22}$$

Equation (22) is exactly the starting point for the path-integral of classical mechanics which was proposed in ref. [1]. If one Fourier-transforms the δ-function by introducing the auxiliary field λ_a and exponentiates the determinant with the ghosts c^a and $\bar{c}_a$, one arrives precisely at the integral (4). This shows that, in a sense, our path-integral can be considered a gauge-fixed version of the "topological action" $S[\phi] = 0$, with Hamilton's equation playing the role of a gauge fixing condition. This turns the local symmetry (19) into the global *BRS*-invariance.

The next question to ask is whether the path-integral of classical mechanics can be used to compute topological invariants. Let us first clarify the meaning of "topological" in the present context. For the kind of topological field theory discussed by Witten [6], the expectation value $\langle\mathcal{O}\rangle$ of some operator $\mathcal{O}$ is a topological invariant if and only if it is invariant under infinitesimal changes in the metric $g_{\mu\nu}$ of the Riemannian manifold on which the theory is defined. Clearly, since we are working on a symplectic manifold (phase-space), which in general does not have a Riemannian structure, the topological invariance of some expectation value must have a different meaning. In our case a quantity is of topological nature if it does not depend on the dynamics given by the Hamiltonian H, *i.e.*, it does not change if we perform small deformations of H or of the associated vector field h.*

Let us change the Hamiltonian vector field h^a by an amount

$$\delta_\eta h^a(\phi^b) \equiv \eta^a(\phi^b) \tag{23}$$

Then eq. (17) and (18) show that $\widetilde{\mathcal{L}}$ changes by a *BRS*-variation

$$\delta_\eta\widetilde{\mathcal{L}} = i[Q, \bar{c}_a\eta^a] \tag{24}$$

This statement is analogous to the property of the theories in ref. [7] that the metric variation of the Lagrangian (the energy-momentum tensor) is a *BRS* commutator. Now

* Deforming h means either deforming ω or H or both.

we investigate under which conditions the expectation value

$$\langle \mathcal{O}(t)\rangle = \int \mathcal{D}\phi^a(t)\cdots \mathcal{O}(\phi(t),\cdots)\ exp\ i\int_{-T}^{T} dt\widetilde{\mathcal{L}} \tag{25}$$

of some local operator $\mathcal{O} = \mathcal{O}(\phi^a(t),\lambda_a(t),c^a(t),\bar{c}_a(t))$ is invariant under (23): $\delta_\eta\langle\mathcal{O}(t)\rangle = 0$. The integral (25) is over fields defined in $[-T,T]$. It includes the integration over all fields at $t=-T$ and over ϕ^a and c^a at $t=+T$. Note that these boundary conditions are *BRS* invariant. Because of the *BRS*-invariance of both the action and the measure, we can prove in the usual way [7] that the expectation value of any *BRS* variation vanishes

$$\langle [Q,\mathcal{O}]\rangle = 0 \tag{26}$$

Exploiting this fact, it is easy to see that the η-variation of $\langle\mathcal{O}(t)\rangle$ is given by

$$\delta_\eta\langle\mathcal{O}(t)\rangle = \langle\delta_\eta\mathcal{O}(t)\rangle + \left\langle [Q,\mathcal{O}(t)]\int dt\ \bar{c}_a\eta^a\right\rangle \tag{27}$$

Again using (26) we see that $\langle\mathcal{O}(t)\rangle$ is a (non-zero) topological invariant, if we require that

$$\begin{aligned}[Q,\mathcal{O}(t)] &= 0\\ \mathcal{O}(t) &\neq [Q,(\cdot)]\\ \delta_\eta\mathcal{O}(t) &= [Q,(\cdot)]\end{aligned} \tag{28}$$

These conditions are very similar to the analogous requirements for topological Yang-Mills theory or the topological σ-model, say. The observable $\mathcal{O}$ must be *BRS*-closed, but not exact, and the η-variation (or the $g_{\mu\nu}$-variation) of $\mathcal{O}$ has to be a BRS-commutator. If these requirements are met, $\langle\mathcal{O}\rangle$ does not depend on the details of the dynamics prescribed by H.

As a first example, let us consider the simplest operator satisfying (28), namely the unit operator $\mathcal{O}=1$. Computing its expectation value with free boundary conditions as in (25) we would obtain $\langle\mathcal{O}\rangle = 0$ because of the ghosts' zero-modes.[‡] This is not

‡ A way out is to insert a "gauge"-fixing function $N(\lambda(0),c(0),\bar{c}(0))$ for these zero modes, as we did in the appendix of the second paper of ref. [1] .

so for periodic boundary conditions (pbc)[♮] , however. Requiring $\phi^a(0)=\phi^a(T)$ and $c^a(0)=c^a(T)$ for some fixed $T>0$, we are led to

$$\begin{aligned}\mathbf{Z}_{pbc} &= \int_{pbc} \mathcal{D}\phi^a(t)\cdots exp\ i\int_0^T dt'\widetilde{\mathcal{L}}\\ &= \int d^{2n}\phi_0\ d^{2n}c_0\ K(\phi_0^a\ c_0^a,T|\phi_0^a,c_0^a,0)\end{aligned} \tag{29}$$

where we used eq.(4) in the second line. Periodic boundary conditions indeed respect *BRS*-symmetry. Under the transformation (12), the first condition $\phi^a(0)=\phi^a(T)$, for instance, becomes $\phi^a(0)+\epsilon c^a(0) = \phi^a(T)+\epsilon c^a(T)$ which is fulfilled since c^a is also periodic. On the other hand, antiperiodic boundary conditions for the ghosts would break *BRS* invariance . We now compute $\mathbf{Z}_{pbc}$ in the limit $T\to 0$ and show later on that $\mathbf{Z}_{pbc}$ actually does not depend on T. In equation (7) we expressed the kernel K as a product of δ- functions involving the solutions $\phi^a_{cl}(t)$ and $c^a_{cl}(t)$ of the equations of motion (1) and (8) . For $T\to 0$ and with initial conditions $\phi^a_{cl}(t=0)=\phi^a_0$, $c^a_{cl}(t=0)=c^a_0$ their solutions simply read

$$\begin{aligned}\phi^a_{cl}(T) &= \phi^a_0 + h^a(\phi_0)T + O(T^2)\\ c^a_{cl}(T) &= c^a_0 + \partial_b h^a(\phi_0)c^b_0 T + O(T^2)\end{aligned} \tag{30}$$

Inserting this into eq. (7) we find for $T\to 0$:

$$\begin{aligned}K(\phi^a_0,c^a_0,T|\phi^a_0,c^a_0,0) &= \delta^{(2n)}(h^a(\phi_0)T)\delta^{(2n)}(\partial_b h^a(\phi_0)c^b T)\\ &= \delta^{(2n)}(h^a(\phi_0))\delta^{(2n)}(\partial_b h^a(\phi_0)c^b_0)\\ &= \delta^{(2n)}(h^a(\phi_0))\ det[\partial_b h^a(\phi_0)]\delta^{(2n)}(c^a_0)\end{aligned} \tag{31}$$

As is to be expected, for closed paths and short times T the kernel K receives contributions only from the points where the vector field $h^a\equiv\omega^{ab}\partial_b H$ vanishes,*i.e.*, for the critical points of the Hamiltonian H. Let us assume for simplicity that H has only isolated, non-degenerate[∘] critical points at $\phi^a_{(p)}$. We then have

$$K(\phi^a_0,c^a_0,T|\phi^a_0,c^a_0,0) = \delta^{(2n)}(c^a_0)\sum_{(p)}\frac{det[\partial_b h^a(\phi_{(p)})]}{|det[\partial_b h^a(\phi_{(p)})]|}\delta^{(2n)}(\phi^a_0-\phi^a_{(p)}) \tag{32}$$

♮ This corresponds to a choice of a particular N function.
∘ This restriction is not essential. Along the lines of ref. [12] one could generalise the discussion to include degenerate Morse theory.

where the sum extends over all critical points in $\mathcal{M}_{2n}$. Using (32) in (29) one finds

$$\begin{aligned} \mathbf{Z}_{pbc} &= \sum_{(p)} \frac{det[\partial_b h^a(\phi_{(p)})]}{|det[\partial_b h^a(\phi_{(p)})]|} \\ &= \sum_{(p)} \frac{det \partial_a \partial_b H(\phi_{(p)})]}{|det[\partial_a \partial_b H(\phi_{(p)})]|} \equiv \sum_{(p)} (-1)^{i_p} \end{aligned} \tag{33}$$

Here i_p denotes the index of the critical point (p), *i.e.*, the number of negative eigenvalues of the Hessian of H at (p) (which in local coordinates has components $\partial_a \partial_b H(\phi^a_{(p)})$). The above result for $\mathbf{Z}_{pbc}$ has a well-known interpretation [13] : it is the Morse theory representation of the Euler number χ of the manifold on which the "Morse function" H is defined. Thus we have found that

$$\mathbf{Z}_{pbc} = \chi(\mathcal{M}_{2n}) \tag{34}$$

This answer is indeed independent of the dynamics of the system under consideration. The Hamiltonian acts only as a Morse function and hence $\mathbf{Z}_{pbc}$ contains only information about the topology of phase-space, but not about the dynamics generated by H. Next we show that eq. (34) holds for any $T > 0$ and not only for $T \to 0$. The argument we use is quite standard in the discussion of the Witten index of supersymmetric quantum mechanics [14] [15] . Recalling eq.(9) we may write eq.(29) as

$$\mathbf{Z}_{pbc} = Tr\left[(-)^F e^{-iT\widetilde{\mathcal{H}}}\right] \tag{35}$$

Here the trace "Tr" is performed over a complete set of functions $\tilde{\varrho}_n(\phi^a, c^a)$ or, equivalently, a set of antisymmetric tensor fields $\varrho^{(p)}_{n,a_1,\cdots a_p}(\phi)$,$p = 0, 1, \cdots, 2n$, which enter the expansion (3). The Hamiltonian $\widetilde{\mathcal{H}}$ acts on them like the Lie-derivative along h. The "Fermion number" operator F counts the degree of the respective differential form, hence $(-)^F = +1(-1)$ for p even (odd).* To give a meaning to eq. (35), we have to introduce a scalar product for forms. This is done by choosing an arbitrary Riemannian metric g_{ab} on $\mathcal{M}_{2n}$. Furthermore, we decide to evaluate the trace in a basis of eigenfunctions of the Laplacian $\Delta_g \equiv d\delta + \delta d$ constructed from the metric g_{ab} $^\nabla$. Let $\{\psi_i^{(p)}\}$

* The factor $(-1)^F$ appears because, due to the ghosts, the basis functions $\tilde{\varrho}_n$ do not necessarily commute.

∇ Here $d(\delta)$ denotes the exterior (co)derivative.

be a complete set of normalized eigenfunctions. Then

$$Tr\left[(-)^F e^{-iT\widetilde{\mathcal{H}}}\right] = \sum_{p=0}^{2n} (-1)^p \sum_i \left\langle \psi_i^{(p)} \middle| e^{-Tl_h} \middle| \psi_i^{(p)} \right\rangle \tag{36}$$

where the inner product $\langle \cdot | | \cdot \rangle$ refers to g_{ab}, and where $\Delta_g \psi_i^{(p)} = \lambda_i^{(p)} \psi_i^{(p)}$. The essential observation [14] is that (36) receives contributions only from the ψ with $\lambda = 0$, because for $\lambda \neq 0$, we have always a "super multiplet" $(\psi, \tilde{\psi})$ of eigenfunctions with the same λ, but a different value of $(-1)^F$:

$$\begin{aligned} \tilde{\psi} &= \frac{1}{\sqrt{\lambda}} (d + \delta) \psi \\ \psi &= \frac{1}{\sqrt{\lambda}} (d + \delta) \tilde{\psi} \end{aligned} \tag{37}$$

Since $exp(-Tl_h)$ commutes with $d + \delta$, we have that

$$\langle \psi | exp(-Tl_h) | \psi \rangle = \left\langle \tilde{\psi} \middle| exp(-Tl_h) \middle| \tilde{\psi} \right\rangle \tag{38}$$

and therefore all contributions to (36) with $\lambda \neq 0$ cancel pairwise. Thus the RHS of (36) reduces to a sum over harmonic forms or, equivalently, to a trace in the de Rham cohomology group $\mathbf{H}^p(\mathcal{M}_{2n}, \Re)$:

$$\mathbf{Z}_{pbc} = \sum_{p=0}^{2n} (-1)^p Tr_{\mathbf{H}^p}\left[e^{-Tl_h}\right] \tag{39}$$

The RHS of eq. (39) has a well-known interpretation in terms of the Lefschetz coincidence theorem [13] , namely it is the Lefschetz number $Lef[exp(-Tl_h)]$ of the mapping induced by $exp(-Tl_h)$ on phase space. The general Lefschets theorem, for an arbitrary mapping K of some manifold into itself, expresses alternating sums like (39) in terms of local data of the fixed point set of the respective mapping. It can be shown that $Lef[K]$ is always integer, and that it does not depend on the Riemannian metric chosen. Furthermore, $Lef[K]$ is a homotopic invariant of K. This implies that, if K is homotopic to the identity map, $K \sim id$, the traces of $\mathbf{H}^p$ are simply given by the Betti numbers b^p:

$$Tr_{\mathbf{H}^p}[K] = Tr_{\mathbf{H}^p}[id] = dim\mathbf{H}^p(\mathcal{M}_{2n}, \Re) = b^p \tag{40}$$

In our case $K \equiv exp(-Tl_h)$ is generated by a continuous time evolution and K is in fact homotopic to the identity (with T playing the role of the homotopy parameter).

Therefore eq.(39) becomes $\mathbf{Z}_{pbc} = \sum_{p=0}^{2n}(-1)^p b^p = \chi(\mathcal{M}_{2n})$ which coincides with the result (34) of the explicit evaluation of the classical path-integral. Since $K \sim id$ for any value of T, the relation above shows that $\mathbf{Z}_{pbc}$ does not depend on T.

The reader will have realized that our classical path-integral has many features in common with the partition function (with pbc) of supersymmetric quantum mechanics on a curved manifold [14], [16]. In fact, both path-integrals evaluate the "Witten index" $Tr[(-1)^F]$. There are also crucial differences, however. From the quantum mechanical point of view, the underlying manifold $\mathcal{M}_{2n}$ is a *configuration space*, *i.e.*, to be able to "quantize" a particle in this space we need a *Riemannian structure* in order to write down a Schrödinger equation or a quadratic term $\frac{1}{2}g_{ab}(\phi)\dot\phi^a\dot\phi^b$ in the action. On the other hand, from the classical mechanics point of view, $\mathcal{M}_{2n}$ is a *phase-space*, *i.e.* we have to require a *symplectic structure* in order to write down Hamilton's equation of motion. At first sight it seems surprising that in evaluating equation (29) it does not matter whether we use the classical or the quantum mechanical propagation kernel K. In both cases K is given by an integral over the space of (based) loops of length T, with different actions, however. Since for the particular observable $\mathbf{Z}_{pbc}$ the value of T may be chosen freely, we can take $T \to 0$, in which case the loops can effectively be identified with the points of $\mathcal{M}_{2n}$. In this situation the form of the action in loop-space does not matter anymore and, in particular, quantum effects are irrelevant.

Now let us try to find more general topological invariants $\langle \mathcal{O} \rangle$. Assuming that $\mathcal{O}$ does not depend on H, eq.(28) requires that $\mathcal{O}$ has a vanishing commutator with Q, but cannot be written as the BRS variation of something. Operators with this property are in one-to-one correspondence with representatives $F^{(p)} \equiv F^{(p)}_{a_1\cdots a_p}d\phi^{a_1}\wedge\cdots\wedge\phi^{a_p}$ of the cohomology classes $\mathbf{H}^p(\mathcal{M}_{2n},\Re)$. If we define $\mathcal{O}^{(0)}_F(t) = F^{(p)}_{a_1\cdots a_p}(\phi(t))c^{a_1}(t)c^{a_2}(t)\cdots c^{a_p}(t)$, it easily follows from the operatorial rules* derived in ref. [1] that $\mathcal{O}^{(0)}_F$ is BRS-closed, but not exact. If $\left\langle \mathcal{O}^{(0)}_F(t)\right\rangle$ is to be a topological invariant, it may not depend on t. This would be guaranteed if $\dot{\mathcal{O}}^{(0)}_F$ were a BRS-commutator. It is easy to verify that this is indeed the case:

$$\dot{\mathcal{O}}^{(0)}_F(t) = [Q, \mathcal{O}^{(1)}_F(t)] \tag{41}$$

with

$$\mathcal{O}^{(1)}_F(t) = pF_{a_1\cdots a_p}(\phi(t))\dot\phi^{a_1}(t)c^{a_2}(t)\cdots c^{a_p}(t) \tag{42}$$

* Recall that $\left[Q, \hat F^{(p)}\right] = (dF^{(p)})\hat{}$, where the $\hat{}$ means that $d\phi^a$ is replaced by c^a

The observables $\mathcal{O}^{(0)}_F$ and $\mathcal{O}^{(1)}_F$ are analogous to Witten's [14] operators W_0 and W_1 out of which the Donaldson invariants are constructed. Since in our case the "base manifold" is just the interval $[-T,T]$, or rather, for $T\to\infty$, the real line R, there are only two classes of such operators. (On a four-manifold the above construction continues and one arrives at five operators $W_0, W_1, \cdots, W_4$.) In our theory we cannot replace more than one c^a by a derivative of ϕ^a. Now let us look at the expectation values $\left\langle \mathcal{O}^{(0)}_F\right\rangle$ and $\left\langle \mathcal{O}^{(1)}_F\right\rangle$. Here we have to distinguish carefully between the path-integral defined on the interval $[-T,+T]$ where the fields ϕ^a and c^a are integrated independently at $t=\pm T$, and the path-integral for a compactified base manifold S^1 where the points $(\phi^a(-T), c^a(-T))$ and $(\phi^a(+T), c^a(+T))$ are identified so that there is only one integration over $\phi(-T)=\phi(T)$ and $c^a(T) = c^a(-T)$. This compactification to S^1 has been used for supersymmetric quantum mechanics [12]. For the first type of boundary condition and the operators $\mathcal{O}^{(0)}_F(t=0)$, we find for the path-integral (25)

$$\begin{aligned}\left\langle \mathcal{O}^{(0)}_F(0)\right\rangle &= \int dx(T)\, dx(0)\, dx(-T)\; K(x(T),T|x(0),0)\; \mathcal{O}^{(0)}_F(x(0)) \\ &\quad \cdot K(x(0),0|x(-T),-T) \\ &= \int dx(0)\; \mathcal{O}^{(0)}_F(x(0))\end{aligned} \tag{43}$$

where we have used the notation $x \equiv (\phi^a, c^a)$ and eq.(4) for $[-T,0]$ and $[0,T]$ respectively.† Obviously $\left\langle \mathcal{O}^{(0)}_F\right\rangle$ does not give rise to interesting topological invariants since for $p\neq 2n$ this expectation value vanishes because of the ghost integration. For $p=2n$ one obtains $\int d^{2n}\phi\; F_{a_1\cdots a_{2n}}c^{a_1\cdots a_{2n}}$ which is proportional to the volume of phase-space (if $b^0 = b^{2n} = 1$). We stress, however, that on $[-T,+T]$ it is possible in principle to have a non-zero expectation value of an operator like $\mathcal{O}^{(0)}_p$ with *non-vanishing ghost charge*. The reason is that, as in any phase-space path-integral over some interval $[-T,T]$ [17] there is a mismatch between the position c^a and the momentum $\bar c_a$ integration: including the position integrations at $t=\pm T$, there is one more position integration than momentum integration. This means that the measure $\mathcal{D}c\;\mathcal{D}\bar c$ is not neutral under the ghost charge transformation*. Later on we shall exploit this fact when we give a path-integral representation of the Maslov index. The situation is different if we compactify $[-T,T]$ to the circle S^1, *i.e.*, if we integrate over closed paths only. Then the c and $\bar c$-integrations match and only operators of vanishing ghost number can have a non-zero expectation

† Note that $\int dx_f\; K(x_f,t_f|x_i,t_i) = \int dx_i\; K(x_f,t_f|x_i,t_i) = 1$.
* Recall from ref. [1] that $(c^a, \bar c_a, \phi^a, \lambda_a)$ has ghost number (1,-1,0,0).

value‡. Since both $\mathcal{O}_F^{(0)}$ and $\mathcal{O}_F^{(1)}$ contain only c's, but no $\bar{c}$'s, no further invariants beyond $\mathbf{Z}_{pbc}$ can be computed in this way. Note also that when working with S^1 we can follow Witten's procedure of integrating $\mathcal{O}_F^{(1)}$ over a 1-cycle in the base manifold in order to obtain a *BRS*-invariant quantity. In fact, using (41), we see that $\oint dt\mathcal{O}_F^{(1)}(t)$ commutes with Q because of the periodic boundary conditions.

We are now going to discuss another kind of "topological" observables which are sensitive to ω and H ‡‡. This discussion is specific to the path-integral of classical mechanics and we do not know if it has a counterpart in quantum field theory. The point is that up to now, in order to integrate over closed paths, we had to specify this by an explicit boundary condition $\phi^a(0) = \phi^a(T)$. This condition selects those initial points ϕ_0^a for which the vector field h^a, or the Hamiltonian H respectively, admits closed trajectories of length T. Typically there will be only very few such trajectories; in fact, it is known that they form a set of measure zero. For a generic Hamiltonian H, almost all initial conditions $\phi^a(0)$ will give rise to trajectories which never close. There exists a special class of Hamiltonians, however, which has the property that they generate closed orbits in phase-space for any initial point ϕ_0^a. This kind of Hamiltonian naturally appears in integrable systems, for instance. Let us assume we are given a (near-)integrable Hamiltonian system $(\mathcal{M}_{2n}, \omega, H)$ which then possesses invariant tori in some part of phase-space, at least. In this region it is possible to transform from the ϕ^a-coordinates to action-angle variables (θ_i, J_i), $i = 1 \cdots n$. Considering the actions $J_i = J_i(\phi^a)$ as the generators of a family of canonical transformations, their integral curves generate the basic homology cycles on the tori. Starting from a given point $\phi^a(0)$ and following the trajectory induced by J_i according to

$$\dot{\phi}^a = \omega^{ab}\partial_b J_i(\phi) \tag{44}$$

we generate the i-th cycle Γ_i. Along this loop the angle variable θ_i increases from 0 to 2π, while the others $(\theta_j, j \neq i)$ do not change at all. The time needed for one circuit depends on the point $\phi^a(0)$. A given initial point is equivalent to a set of numerical values $\{J_i, i = 1 \cdots n\}$ for the action variables. This set defines a certain n-torus in $\mathcal{M}_{2n}$. The curves $\Gamma_i, i = 1, \cdots, n$ generated by $J_i(\phi^a)$ starting from $\phi^a(0)$ lie on this torus for all t. From the path-integral point of view it is attractive to use $J_i(\phi)$, rather than H, as a Hamiltonian, because for this class of Hamiltonians the path-integral is saturated

‡ This follows from the vanishing index of the kinetic operator of the ghosts. The situation is similar as to that of topological quantum mechanics, see ref. [12].

‡‡ Again we stress the meaning of "topological" (or sensitive) here: it means that they do not "feel" small deformations of ω, $\mathcal{H}$, but only "large" deformations.

by closed paths even in the case when we work on $[-T, T]$ with independent ϕ^a and c^a-integration at $t = \pm T$°. We begin with a very simple example. We take the path-integral (25) over $[-T, T]$ with independent integrations at the end- points and replace the Hamiltonian H by the actions J_i. Furthermore we pick a certain representative $F = F_a d\phi^a$ of $\mathbf{H}^1(\mathcal{M}_{2n}, \Re)$ and define the observable

$$\mathcal{F}(\phi_0^a) = \delta^{(2n)}(\phi^a(0) - \phi_o^a)\delta^{(2n)}(c^a(0)) \oint dt' F_a(\phi^b(t'))\dot{\phi}^a(t') \tag{45}$$

which is similar to eq.(42) for p=1•. $\mathcal{F}$ is a functional of the (closed) path $\phi^a(t')$ and the t'-integration is over a full period. Clearly $\mathcal{F}$ is *BRS*-invariant, since a *BRS*-transformation leads to a homotopic deformation of the path which does not change the integral or the RHS of (45) since dF=0. Moreover, $\phi^a(0)$ would change by an amount $\epsilon c^a(0)$, which is zero due to the second δ- function. If we now compute the expectation value $\langle\mathcal{F}\rangle$ it is easy to see that the path-integral calculates $\oint F_a d\phi^a$ along the classical trajectories ϕ_{cl} generated by J_i starting from ϕ_0^a:

$$\langle\mathcal{F}(\phi_0^a)\rangle = \oint dt' F_a(\phi_{cl}^b(t', \phi_0^b))\dot{\phi}_{cl}^a(t', \phi_0^b) \tag{46}$$

Even this somewhat trivial example nicely illustrates the interplay between dynamics (or topological-)invariance, *BRS*-invariance and homotopic invariance. If F is closed, $\mathcal{F}$ is *BRS*-invariant and therefore $\langle\mathcal{F}\rangle$ is insensitive to the details of the "gauge-fixing", *i.e.*, of the Hamiltonian. On the other hand, F being closed means that $\oint_{\Gamma_i} F$ is invariant under homotopic deformations of Γ_i, and the *BRS*-transformation $\phi^a(t) \to \phi^a(t) + \epsilon c^a(t)$ provides precisely such an infinitesimal transformation. Nevertheless, contrary to $\chi(\mathcal{M}_{2n})$, say, $\langle\mathcal{F}\rangle$ depends on the the Hamiltonian, since replacing J_i by $J_k, k \neq i$, for instance, F is integrated over the cycle Γ_k which yields a different result in general.

Now let us apply the above method to the Maslov index [18] [19] [20]. The Maslov index occurs in two different versions: as the even integer μ_i in the Einstein-Brillouin-Keller semiclassical quantization condition

$$J_i = (n_i + \frac{1}{4}\mu_i)\hbar \ , \ n_i = 0, 1, 2 \cdots \tag{47}$$

for integrable systems, and as the even or odd integer entering the phase-shifts of the WKB-wave functions. Traditionally the Maslov index was calculated by investigating the

° Recall that in this situation we can have $\langle\mathcal{O}\rangle \neq 0$ even if $\mathcal{O}$ contains ghosts.

• From now on t is no longer the physical time, but simply a parameter along the curves generated by the J_i's.

caustics of Lagrangian manifolds, but here we shall follow the approach of Littlejohn [20] , who interpreted μ_i as a winding number in the $Sp(2n)$ group manifold. We consider an integrable system with action variables $J_i, i = 1 \cdots n$, giving rise to a set of basic homology cycles Γ_i on the invariant tori. To each cycle we can associate an integer μ_i by the following construction. Let Γ_i be parametrized by $\phi^a(t)$, $t \in [0, 2\pi]$ which solves eq.(44) for some $\phi^a(t=0) = \phi^a_o$. We define the 2π periodic matrix function

$$S^a_b(t) = \frac{\partial \phi^a(t)}{\partial \phi^b(0)} \tag{48}$$

which describes the change of the position of the particle at time t due to a small change of the initial conditions. Note that this matrix is symplectic, $S^a_b \in Sp(2n)$, since it is the Jacobian matrix of a canonical transformation [5] . Now divide the phase-space coordinates $(\phi^1, \cdots, \phi^{2n})$ into a set of n "positions" q^i and n "momenta" p^i, $i = 1, \cdots, n$, and then form the (n x n) matrices $S^q_q(t) \equiv S^{q_i}_{q_j}(t)$ and $S^q_p(t) \equiv S^{q_i}_{p_j}$. As was shown by Littlejohn and Robbins [20] , these matrices can be used to compute the Maslov index μ_i associated to Γ_i:

$$\begin{aligned} \mu_i &= 2\,W\big[det\{S^q_q + iS^q_p\}\big] \\ &= 2 \oint_{\Gamma_i} \frac{dt}{2\pi i} ln\ det\big[S^q_q(t) + iS^q_p(t)\big] \end{aligned} \tag{49}$$

First of all this formula expresses μ_i as a winding number W in the complex plane: the integral on the RHS of eq.(49) counts the number of times the value of the function $det\big[S^q_q + iS^q_p\big]$ encircles the origin of the complex plane, in a period of oscillation of ϕ. Eq.(49) can be used to assign an integer $W[S]$ to any periodic, symplectic matrix function S. Clearly $W[S]$ can depend only on the homotopy class of S. Since $\prod_1(Sp(2n)) = Z$, these classes are labelled by a single integer (the winding number), which turns out$^\nabla$ to be given by $W(S)$. This means that the winding number of $det[S^q_q + iS^p_q]$ in the complex plane is the same as the winding number of S^a_b on the group manifold of $Sp(2n)$. The formula (49) is not manifestly canonically invariant because of the arbitrary splitting of ϕ^a into positions and momenta. It can be shown, hovever, that μ_i does not depend on this splitting [19] . For more details about the Maslov index, we have to refer to the literature [18] , [19] , [20] , [21] . We now proceed to give a path-integral representation for μ_i. The crucial observation is that the matrix $S^a_b(t)$ of eq.(48) is essentially the same as

∇ For a proof see the appendix of ref. [21] .

our ghost fields. In fact, if $\phi^a(t)$ solves eq.(44), this matrix-function is a solution of

$$\big[\partial_t \delta^a_b - \partial_b j^a_i(\phi(t))\big]\, S^b_c(t) = 0 \tag{50}$$

with the initial condition $S^a_b(0) = \delta^a_b$ (here $j^a_i = \omega^{ab}\partial_b J_i$.) Comparing eqs.(50) and (8) we see that, for each fixed value of the lower index c, S^b_c evolves in the same way as the ghost c^b. This means that an object like S^a_b appears naturally in the N-particle path-integral of eq.(15) if we put $N = 2n$. Using the initial condition $c^a_k(0) = \zeta\delta^a_k$ for the 2n "species" of ghosts, we can recover S^a_b at a later time by $S^a_b(t) = \frac{\partial}{\partial\zeta} c^a_{(b)}(t)$. Here ζ is an anticommuting constant introduced because S^a_b is a commuting object, whereas the ghosts are anticommuting. Now it is easy to express μ_i as an expectation value of the form

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi^a_k(t)\ \mathcal{D}\lambda_{a,k} \cdots \mathcal{O}\ exp\ i \int_{-T}^{T} dt \tilde{\mathcal{L}}^{J_i}_{2n} \tag{51}$$

with $t \in [-T, +T]$,$T \geq 2\pi$, and independent integrations at the end-points† . We define the observables as

$$\begin{aligned} \mathcal{O}(\phi^a_o) &= 2 \prod_{k=1}^{2n} \delta^{(2n)}(\phi^a_k(0) - \phi^a_0) \prod_{k=1}^{2n} \delta^{(2n)}(c^a_k(0) - \zeta\delta^a_k) \\ &\quad \cdot \int_o^{2\pi} \frac{dt}{2\pi i} ln\ det\big[\frac{\partial}{\partial\zeta}[c^q_q(t) + ic^q_p(t)]\big] \end{aligned} \tag{52}$$

The first δ-function fixes a common starting point for all 2n-paths we follow, and the second assigns the initial values to the ghosts. Obviously when computing the expectation value $\langle \mathcal{O}_i(\phi^a_0)\rangle$, the path-integral is saturated by a single classical trajectory on which the observable $\mathcal{O}_i(\phi^a_o)$ is evaluated. This means that $\langle \mathcal{O}(\phi^a_0)\rangle = \mu_i$ for any initial point ϕ^a_0. The observable $\mathcal{O}(\phi^a_0)$ is BRS invariant except for the δ function fixing the starting point. Under the BRS-transformation it is changed by an amount $\epsilon_k c^a_k = \zeta\epsilon^a$‡ . To obtain a fully BRS invariant observable we can omit the δ-function, which fixes the initial point ϕ_0, from (52) or, more precisely, use the operator $\mathcal{O} = \Omega^{-1} \int d^{2n}\phi_0\ \mathcal{O}(\phi_0)$ where Ω is the volume of the phase-space* . Obviously, since all trajectories generated by J_i lead to

† And again we have substituted $H \to J_i$.

‡ Here ϵ_k is the transformation parameter for the "species" k.

* Here we assume that the J_i can be defined everywhere in phase-space. If $\mathcal{M}_{2n}$ is non- compact, one might need a sort of "infra-red regularisation" to define $\mathcal{O}$.

the same Maslov index, we have again $\langle \mathcal{O} \rangle = \mu_i$ but this time with a manifestly *BRS*-invariant operator $\mathcal{O}$. Applying the above arguments, the *BRS*-invariance of $\mathcal{O}$ implies the invariance of μ_i under "small" deformations of J_i, which in turn translates into small deformations of the original Hamiltonian H which gave rise to the actions $\{J_i\}$. Note again that "topological" invariance has a different meaning for $\mathbf{Z}_{phc}$ and for the Maslov index. In the first case "global" properties of only $\mathcal{M}_{2n}$ mattered, whereas in the second case we picked up "global" or ("large") properties of the whole triplet $(\mathcal{M}_{2n}, \omega, H)$ that characterise the dynamical systems. That the Maslov indices characterise the topological properties of spaces bigger[†] than just the phase-space $\mathcal{M}_{2n}$ is confirmed also by the analysis done by Arnol'd[19]. There it was shown that the Maslov index has a deep topological meaning related to the cohomology (with values in the integers) of the infinite-dimensional Grassmannian-Lagrangian consisting of all the Lagrangian surfaces in phase-space. In our analysis, and in the one of ref.[20], the cohomological meaning of the Maslov-index is not obvious. We will come back in the future[22] to a path-integral representation of the Maslov-index in the Arnol'd version. In the same work we will also give a path-integral representation to the invariants related to Lagrangian intersections discovered recently by Floer[23]. Last but not least, we hope to be able to use our path-integral as a KMS-functional[6],[24] to study "topological" properties of the space of *classical-paths* and have indications on the integrability or KAM nature of our system on the lines of ref.[6]. In general we think that our functional approach to classical mechanics can be used to give a path-integral representation for many more "topological" invariants than the two discussed in this introductory work.

We close with a few remarks on the meaning of BRS-invariant boundary conditions. Of course, a priori, classical mechanics is not a topological field theory and the time evolution of some phase-space density $\varrho(\phi)$ or its generalization $\tilde{\varrho}(\phi, c)$ has little to do with topology. The time evolution of these functions is described by the path-integral (4) whose boundary conditions are *not BRS* invariant. Hence the dynamics of $\tilde{\varrho}$ indeed "feels" the way in which we "gauge fix" the symmetry (19), *i.e.*, it does depend on the local form of the Hamiltonian which enters the gauge fixing condition. It is only for the very limited class of BRS-invariant observables together with BRS-invariant boundary conditions that we may obtain a topological invariant. We can rephrase this by looking at the theory in its Schrödinger picture (see ref.[1]). In general we are dealing with functions $\tilde{\varrho}$ which are

† Like for example the space of the triplets $(\mathcal{M}_{2n}, \omega, H)$.

not BRS-invariant. Requiring them to be annihilated by Q,

$$Q\tilde{\varrho} \equiv c^b \partial_b \tilde{\varrho}(\phi^a, c^a) = 0 \tag{53}$$

means that, for 0-forms (for example), the only allowed or "physical" state is $\varrho(\phi) =$ *const*. Clearly, from the usual dynamical point of view, this is a rather uninteresting situation. Similarly, for a general p-form $\varrho_{a_1 \cdots a_p} c^1 \cdots c^p$ eq.(53) selects the closed forms only. If we make the additional requirement that $\tilde{\varrho}$ should be of the form $\varrho \neq Q(\cdot)$, we see that the "physical-state-space" of the classical particle is given by the de Rham cohomology classes $\mathbf{H}^p(\mathcal{M}_{2n}, \Re)$. What is meant by "physical" in the above context, is that in these states the gauge-fixing is not felt. This is the reason why, in gauge-theories, the BRS-condition has to be imposed to get acceptable or "physical" states. However, for (potentially) topological theories, like classical mechanics, the situation is different : by imposing our *BRS*-symmetry, one is able to extract *only* topological information from the theory, like de Rham cohomology classes in our case.
Anyhow we feel[22] that "something" like a physical-state condition should be imposed also on (standard non-topological) CM by using a different *BRS* generator. This generator is the one we get once we study the symmetry of $\mathbf{Z}_{cm}$ with the N-function* inserted. Let us remember that the N-function acts as a "gauge-fixing" for the ghost zero modes and so the expectation values of every observable have to be N-independent : this will modify our original *BRS* charge. Using this different *BRS* charge, the new physical-state condition acquires a meaning once we work in the GNS[25] representation for classical mechanics, and there it ensures the *positivity* of the classical probability[22].

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* This is the function that we introduced in the appendix of the second reference of ref.[1] (and previously in this paper) to cure the problem of the ghost zero-modes.

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